THE CAMERON-MARTIN THEOREM
STAT 206B

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Abstract. In this review paper we will give a presentation of the high-level ideas of [Jan97] and presenting a simplified version of the main results and proofs.

A process which is a Brownian motion with respect to a measure P, will probably not be a Brownian motion with respect to another measure Q. Furthermore, if X and Y are random variables which are independent with respect to P, they may not be independent with respect to another measure Q. In this review paper we study and describe how a Gaussian measure changes under translation by certain elements. Every Gaussian stochastic process defines a space of functions on the index set, which we will call the Cameron-Martin space, it is going to be this space which gives those directions in which translation are going to be 'nice'.

Let $\mathcal{X}$ be a separable Fréchet space. A probability measure γ on $(\mathcal{X}, \mathbb{B}(\mathcal{X}))$ is said to be Gaussian if $\gamma \circ f^{-1}$ is a Gaussian measure in $\mathbb{R}$ for every $f \in \mathcal{X}^*$. We also define the mean a_γ and the covariance K_γ of γ by

$$a_\gamma(f) := \int f(x)\gamma(dx)$$
$$K_\gamma(f,g) := \int |f(x) - a_\gamma(f)||g(x) - a_\gamma(g)|\gamma(dx)$$

Observe that $f \longmapsto a_\gamma(f)$, $(f,g) \longmapsto K_\gamma(f,g)$ are linear and bilinear in $\mathcal{X}^*$ respectively, moreover it can also be proven that a and K are continuous when $\mathcal{X}$ is a Banach space, and the latter is non-negative definite. Furthermore a pair (a, K) determines γ and its characteristic function is given by

$$\hat{\gamma}(f) = \exp\{ia(f) - \frac{1}{2}K(f,f)\} \ \ , f \in \mathcal{X}^*$$

We will say tat γ is centered when $a_\gamma = 0$, in that case the bilinear form K_γ is the restriction of the inner product in $\mathbb{L}_2(\mathcal{X}, \gamma)$ to $\mathcal{X}^*$. Concretely, there is a canonical embedding $I^* : \mathcal{X}^* \hookrightarrow \mathbb{L}_2(\mathcal{X}, \gamma)$, and the closure of $I^*(\mathcal{X})$ in $\mathbb{L}_2(\mathcal{X}, \gamma)$ is called the space of *measurable linear functionals*, denoted by $\mathcal{X}^*_\gamma$.

In the following it will be more convenient to consider I^* as an embedding $I^* : \mathcal{X}^* \hookrightarrow \mathcal{X}^*_\gamma$. The dual operator $I : \mathcal{X}^*_\gamma \hookrightarrow \mathcal{X}$ is defined by a natural relation

$$(f, Iz) = (I^*f, z)_{\mathcal{X}^*_\gamma} = \mathbb{E}(f, X)z(X) \ , \forall \, f \in \mathcal{X}^* \, , \, z \in \mathcal{X}^*_\gamma$$

where X is a Gaussian vector having measure γ. Under usual assumptions it can be proven that this operator exists.

1. The Cameron-Martin space

We now introduce the Cameron-Martin space.

Definition 1.1. *For every $h \in \mathcal{X}$ set*

$$|h|_H := \sup\{f(h) \mid f \in X^* \ , ||I^*(f)||_{\mathbb{L}_2(\mathcal{X},\gamma)} \leq 1\}$$

The Cameron-Martin space is defined by

$$H := \{h \in \mathcal{X} \mid |h|_H < \infty\}$$

Observe that

$$||h||_{\mathcal{X}} = \sup\{f(h) \mid ||f||^*_{\mathcal{X}} \leq 1\} \leq \sup\{f(h) \mid ||I^* f||_{\mathbb{L}_2(\mathcal{X},\gamma)} \leq c\} \leq c|h|_H$$

where c is the norm of I^*, so H is continuously embedded in $\mathcal{X}$. We know that this embedding is moreover compact (for a proof see Theorem 2.4.7 in [Bog15]) and we shall see that the norms $||\cdot||_{\mathcal{X}}$, and $|\cdot|_H$ are not equivalent in H.

Theorem 1.1. *Let γ be a Gaussian measure in a separable Banach space $\mathcal{X}$, and let H be its Cameron-Martin space. The following statements hold*

(i) The unit ball $B^H(0,1)$ of H is relatively compact in $\mathcal{X}$ and hence the embedding $H \hookrightarrow \mathcal{X}$ is compact.
(ii) H is the intersection of all Borel full measure subspaces of $\mathcal{X}$.
(iii) If $\mathcal{X}^*_\gamma$ is infinte dimensional, then $\gamma(H) = 0$

The Cameron-Martin space is a Hilbert space and moreover a reproducible kernel Hilbert space. Notice that if γ is non-degenerate then two different elements of $\mathcal{X}^*$ define two different elements of $\mathcal{X}^*_\gamma$, but if γ is degenerate two different elements of $\mathcal{X}^*$ may define elements coinciding γ-a.e.

We define the operator $R_\gamma : \mathcal{X}^*_\gamma \longmapsto (X^*)'$ by

$$R_\gamma f(g) := \int f(x)[g(x) - a_\gamma(g)]\gamma(dx) \ , \qquad f \in \mathcal{X}^*_\gamma \ , \ g \in \mathcal{X}^*$$

Observe that

$$R_\gamma f(g) = \langle f, g - a_\gamma(g)\rangle_{\mathbb{L}_2(\mathcal{X},\gamma)} \tag{1}$$

It is important to notice that indeed R_γ maps $\mathcal{X}^*_\gamma$ into $\mathcal{X}$, see Theorem 3.2.3 in [Bog15].

Theorem 1.2. *An element $h \in \mathcal{X}$ belongs to H if and only if there is $\hat{h} \in \mathcal{X}^*_\gamma$ such that $h = R_\gamma \hat{h}$. In this case,*

$$|h|_H = ||\hat{h}||_{\mathbb{L}_2(\mathcal{X},\gamma)}$$

Therefore $R_\gamma : X^*_\gamma \longmapsto H$ is an isometry and H is a Hilbert space with the inner product

$$[R_\gamma h, R_\gamma k]_H := \langle \hat{h}, \hat{k}\rangle_{\mathbb{L}_2(\mathcal{X},\gamma)}$$

Proof. If $|h|_H < \infty$, we define the map $L : I * (\mathcal{X}^*) \longmapsto \mathbb{R}$ setting

$$L(I^* f) := f(h) \ \ \forall \ f \in \mathcal{X}^*$$

Such map is well defined since the estimate

$$|f(h)| \leq ||I^* f||_{\mathbb{L}_2(\mathcal{X},\gamma)}|h|_H \tag{2}$$

implies that if $I^* f_1 = I^* f_2$, then $f_1(h) = f_2(h)$. The map L is also continuous with respect to the $\mathbb{L}_2$ topology by the previous estimate (2). Then L can be continuously extended to X^*_γ; by the Riesz representation theorem there is a unique $\hat{h} \in \mathcal{X}_\gamma^*$ such that the extension (which we still denote by L) is given by

$$L(\phi) = \int \phi(x)\hat{h}(x)\gamma(dx) \ \ \forall \phi \in \mathcal{X}_\gamma^*$$

In particular, for any $f \in \mathcal{X}^*$,

$$f(h) = L(I^* f) = \int I^* f(x)\hat{h}(x)\gamma(dx) = f(R_\gamma \hat{h})$$

therefore $R_\gamma \hat{h} = h$ and

$$|h|_H = \sup\{f(h) \mid f \in \mathcal{X}^* \ , \ ||I^* f||_{|} f(h)| \leq ||I^* f||_{\mathbb{L}_2(\mathcal{X},\gamma)} \leq 1\} = ||\hat{h}||_{|} f(h)| \leq ||I^* f||_{\mathbb{L}_2(\mathcal{X},\gamma)} |h|_H$$

Conversely, if $h = R_\gamma \hat{h}$, then by (1) for all $f \in \mathcal{X}^*$ we have

$$f(h) = f(R_\gamma \hat{h}) = \int I^* f(x)\hat{h}(x)\gamma(dx) \leq ||\hat{h}||_{\mathbb{L}_2(\mathcal{X},\gamma)} ||I^* f||_{\mathbb{L}_2(\mathcal{X},\gamma)}$$

whence $|h|_H < \infty$. □

As the space $\mathbb{L}_2(\mathcal{X}, \gamma)$ is separable, hence $\mathcal{X}_\gamma^*$ being a subspace is also separable. Therefore, H, being isometric to a separable space is separable.

Remark 1.1. *An alternative approach to kernel's construction is known as a concept of* Reproducing Kernel *or* Reproducing Kernel Hilbert Space. *Let T be an arbitrary set, and let $K : T \times T \longmapsto \mathbb{R}$ be a non-negative definite function (called a kernel). The space Hreproducing the kernel K is a class of functions $f : T \longmapsto \mathbb{R}$ .It is constructed as follows. We take a linear span of functions $K(t, \cdot)$, $t \in T$,and introduce a scalar product by*

$$\langle K(s, \cdot), K(t, \cdot)\rangle := K(s, t) \ , \quad s, t, \in T$$

Then H is the completion of this span with respect to the Hilbert distance. It can be proven (see section 4.3 in [Lif12]) that both constructions coincide.

The main importance of the Cameron-Martin space is that it characterises precisely those directions in which translations leave the measure 'quasi-invariant' in the sense that the translated measure has the same null sets as the original measure.

We finish this section by constructing the classical Cameron-Martin space, but we first need a factorization result[1].

Theorem 1.3. *Let $\mathcal{H}$ be a Hilbert space and let γ be a Gaussian measure with covariance operator K. Let $J : \mathcal{H} \longmapsto \mathcal{H}$ be an injective linear mapping such that factorization*

$$K = JJ^*$$

holds. Then the Cameron-Martin space can be expressed as $H = J(\mathcal{H})$, while the scalar product and norm in H admit representations

$$\langle h_1, h_2\rangle_H = \langle J^{-1}h_1, J^{-1}h_2\rangle_{\mathcal{X}}$$
$$|h|_H = ||J^{-1}h||_{\mathcal{X}}$$

for all $h_1, h_2 \in H$.

[1] For a proof we refer to Theorem 4.1 in [Lif12]

Example 1.1. *Let* $\mathcal{X} = C[0,1]$ *and let* W *be a Wiener process with measure* γ *and* $\mathcal{H} = \mathbb{L}_2[0,1]$*. We define the operator* $J : \mathcal{H} \longmapsto \mathcal{H}$ *as*

$$(J\ell)(t) = \int_0^t \ell(s)ds$$

Recall, by Riesz-Markov-Kakutani, that $\mathcal{X}^* = \mathbb{M}[0,1]$ *is a space of (sign measures) on* $[0,1]$*, and* $J^* : \mathbb{M}[0,1] \longmapsto \mathcal{H}$ *is given by*

$$(J^*\mu)(s) = \mu[s,1]$$

Then

$$\begin{aligned}(JJ^*\mu)(t) &= \int_0^t (J^*\mu)(s)ds = \int_0^t \mu[s,1]ds \\ &= \int_0^1 \int_0^1 \mathbb{1}_{s\leq t}\mathbb{1}_{s\leq u}\mu(du)ds \\ &= \int_0^1 \min\{t,u\}\mu(du) = K\mu(t)\end{aligned}$$

By the Factorization Theorem (1.3), the Cameron-Martin space for Wiener measure is given by

$$H = \left\{ h \mid h(t) = \int_0^t \ell(s)ds, \ell \in \mathbb{L}_2[0,1] \right\} = \{h \in AC[0,1], h(0) = 0, h' \in \mathbb{L}_2[0,1]\}$$

where AC *denotes the class of absolutely continuous functions. The norm and scalar product are given by*

$$\begin{aligned}|h|_H^2 &= \int_0^1 h'(s)ds \\ (h_1,h_2)_H &= \int_0^1 h_1'(s)h_2'(s)ds\end{aligned}$$

2. The Cameron-Martin Theorem

In this section we will present Theorem 14.17 in [Jan97]. Instead of following the method of proof given there, we follow [Bog15; Hai09; MR], where the main tool is Hellinger's Theorem. For a proof we refer to Proposition 2.12.6 in [Bog15].

Theorem 2.1 (Hellinger's Theorem)**.** *Let* μ *and* ν *be two probability measures on a measurable space* $(\Omega, \mathcal{B})$*, and let* λ *be a measure on* $\mathcal{B}$ *such that* $\mu \ll \lambda$*, and* $\nu \ll \lambda$*. Then the number*

$$H(\mu,\nu) := \int \sqrt{\frac{d\mu}{d\lambda}}\sqrt{\frac{d\nu}{d\lambda}}d\lambda$$

is independent of λ *and the following inequalities hold true:*

$$2[1 - H(\mu,\nu)] \leq ||\mu - \nu|| \leq 2\sqrt{1 - H(\mu,\nu)^2}$$

where $||\mu - \nu|| = ||\frac{d\mu}{d\lambda} - \frac{d\nu}{d\lambda}||_{\mathbb{L}_1(\lambda)}$ *is the variation distance.*

Hellinger's theorem gives us a nice characterization of mutually singular measures. Note that $||\mu-\nu|| = 2$ if and only if $H(\mu,\nu) = 0$. Let us show that this is equivalent to $\mu \perp \nu$. Now, $H(\mu,\nu) = 0$ if and only if the set $A = \{\frac{d\mu}{d\lambda}\frac{d\nu}{d\lambda} \neq 0\}$ verifies $\lambda(A) = 0$, hence also $\mu(A) = \nu(A) = 0$. Therefore the measurable set $E = \{\frac{d\mu}{d\lambda} = 0\,, \frac{d\nu}{d\lambda} > 0\}$ we have $\mu(E) = \nu(E^c) = 0$. Then we have the following lemma

Lemma 2.2. $\mu \perp \nu$ *if and only if* $H(\mu,\nu) = 0$*, if and only if* $||\mu - \nu|| = 2$*, for* μ,ν *probability measures.*

Let γ be a measure and $h \in \mathcal{X}^*$; the distribution γ_h defined by

$$\gamma_h(B) = \gamma(B - h)$$

is called a shift of γ by h. We are interested in checking the absolute continuity of γ_h with respect to γ. If $\gamma_h \ll \gamma$ then h is called admissible shift for γ. If ch is an admissible shift for γ for all $c \in \mathbb{R}$, then we say that h defines an admissible direction for γ.

We are interested in the case of Gaussian measures. One can essentially observe what is going on in the finite dimensional case.

Example 2.1. *Consider $\mathcal{X} = \mathbb{R}^d$. If $\gamma = N(a, Q)$, then for $f \in \mathbb{R}^d$ we have*

$$||I^* f||_{\mathbb{L}_2(\mathbb{R}^d, \gamma)} = \int \langle x - a, f\rangle^2 N(a, Q)(dx) = \langle Qf, f\rangle$$

and therefore $|h|_H$ is finite if and only if $h \in Q(\mathbb{R}^d)$, so $H = Q(\mathbb{R}^d)$ is the range of Q. If Q^{-1} is invertible, $h = R_\gamma \hat{h}$ if and only if $\hat{h}(x) = \langle Q^{-1}h, x\rangle$. Moreover, if γ is non-degenerate the measures γ_h defined by $\gamma_h(B) = \gamma(B - h)$ are all equivalent to γ, i.e. $\gamma \ll \gamma_h$, and $\gamma_h \ll \gamma$, and $\gamma_h = \varrho_h \gamma$ with

$$\varrho_h := \exp\{\langle Q^{-1}h, x\rangle - \frac{1}{2}|h|^2\} = \exp\{\hat{h}(x) - \frac{1}{2}|h|^2\}$$

In the (general) infinite dimensional case, not every shift is an admissible one but for admissible shifts the form of the density is exactly the same: it is an exponent of linear functional multiplied by a normalizing quadratic constant. We start with a preliminary result.

Lemma 2.3. *For any $g \in \mathcal{X}^*_\gamma$, the measure*

$$\mu_g = \exp\{g - \frac{1}{2}||g||^2_{\mathbb{L}_2(\mathcal{X},\gamma)}\}\gamma$$

is a Gaussian measure with characteristic function

$$\hat{\mu}_g(f) = \exp\{if(R_\gamma g) + ia_\gamma(f) - \frac{1}{2}||I^* f||^2_{\mathbb{L}_2(\mathcal{X},\gamma)}\}$$

Proof. Notice that the image of γ under the measurable function g is still a Gaussian measure given by $N(0, ||g||^2_{\mathbb{L}_2(\mathcal{X},\gamma)})$. Then

$$\int \exp|g(x)|\gamma(dx) = \int e^{|t|} N(0, ||g||^2_{\mathbb{L}_2(\mathcal{X},\gamma)})(dt) < \infty$$

hence $\exp|g| \in \mathbb{L}_1(\mathcal{X}, \gamma)$ and μ_g is a finite measure. Moreover μ_g is a probability measure since

$$\mu_g(\mathcal{X}) = \int \exp\{g - \frac{1}{2}||g||^2_{\mathbb{L}_2(\mathcal{X},\gamma)}\}\gamma(dx) = \exp\{-\frac{1}{2}||g||^2_{\mathbb{L}_2(\mathcal{X},\gamma)}\}\int e^t N(0, ||g||^2_{\mathbb{L}_2(\mathcal{X},\gamma)})(dt) = 1$$

Now,

$$\begin{aligned}
\exp\{-\frac{1}{2}||g||^2_{\mathbb{L}_2(\mathcal{X},\gamma)}\}\int \exp\{i(f(x) - tg(x))\}\gamma(dx) &= \exp\{-\frac{1}{2}||g||^2_{\mathbb{L}_2(\mathcal{X},\gamma)}\}\hat{\gamma}(f - tg) \\
&= \exp\{-\frac{1}{2}||g||^2_{\mathbb{L}_2(\mathcal{X},\gamma)}\}\exp\{ia_\gamma(f - tg) - \frac{1}{2}||I^*(f - tg)||^2_{\mathbb{L}_2(\mathcal{X},\gamma)}\} \\
&= \exp\left\{tf(R_\gamma g) - \frac{1 - t^2}{2}||g||^2_{\mathbb{L}_2(\mathcal{X},\gamma)} + ia_\gamma(f) - \frac{1}{2}||I^* f||^2_{\mathbb{L}_2(\mathcal{X},\gamma)}\right\}
\end{aligned}$$

Now, the entire holomorphic functions

$$z \longmapsto \exp\{-\frac{1}{2}||g||^2_{\mathbb{L}_2(\mathcal{X},\gamma)}\} \int \exp\{i(f(x) - zg(x))\}\gamma(dx)$$

$$z \longmapsto \exp\left\{zf(R_\gamma g) - \frac{1-z^2}{2}||g||^2_{\mathbb{L}_2(\mathcal{X},\gamma)} + ia_\gamma(f) - \frac{1}{2}||I^* f||^2_{\mathbb{L}_2(\mathcal{X},\gamma)}\right\}$$

coincide for $z \in \mathbb{R}$, and so they coincide for all $\mathbb{C}$. So with $z = i$ we get the desired formula. □

Theorem 2.4 (Cameron-Martin Theorem[2]). *For $h \in \mathcal{X}$, denote the shift by γ_h. If $h \in H$ the measure $\gamma : h$ is equivalent to γ and $\gamma_h = \varrho_h \gamma$, with*

$$\varrho_h(x) := \exp\{\hat{h}(x) - \frac{1}{2}|h|^2_H\} \tag{3}$$

where $\hat{h} = R_\gamma^{-1} h$. If $h \in H$ then $\gamma_h \perp \gamma$. Hence $\gamma_h \ll \gamma \ll \gamma_h$ if and only if $h \in H$.

The density formula ϱ_h is referred as the Cameron-Martin formula.

Proof. For $h \in H$, let us compute the characteristic function of γ_h. For any $f \in \mathcal{X}^*$ we have

$$\begin{aligned}\hat{\gamma}_h(f) &= \int \exp\{if(x)\}\gamma_h(dx) = \int \exp\{if(x+h)\}\gamma(dx) \\ &= \exp\{if(R_\gamma \hat{h}) + ia_\gamma(f) - \frac{1}{2}||I^* f||^2_{\mathbb{L}_2(\mathcal{X},\gamma)}\}\end{aligned}$$

Then by Lemma (2.3) we get that $\gamma_h = \varrho_h \gamma$, where the density is given by the Cameron-Martin formula (3).

Now, we prove that if $h \notin H$ then $\gamma_h \perp \gamma$.For this, we first consider the 1-dimensional case. If γ is a Dirac measure in $\mathbb{R}$, then $\gamma_h \perp \gamma$ for any $h \neq 0$, and $||\gamma - \gamma_h|| = 2$. Otherwise, if $\gamma = N(a, \sigma^2)$ is a non-degenerate Gaussian measure in $\mathbb{R}$, then $\gamma : h \ll \gamma$ with $\frac{d\gamma_h}{d\gamma}(t) = \exp\{-\frac{h}{2\sigma^2}\frac{h(t-a)}{\sigma^2}\}$.

Now,

$$\begin{aligned}H(\gamma, \gamma_h) &= \int \sqrt{\frac{d\gamma}{d\gamma}(x) + \frac{d\gamma_h}{d\gamma}(x)}\gamma(dx) \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int \exp\{-\frac{h^2}{4\sigma^2} + \frac{h(x-a)}{2\sigma^2} - \frac{(x-a)^2}{2\sigma^2}\}dx \\ &= \frac{1}{\sigma\sqrt{2\pi}} \exp\{-\frac{h^2}{4\sigma^2}\} \int \exp\{\frac{ht - t^2}{2\sigma^2}\}dt \\ &= \exp\{-\frac{h^2}{8\sigma^2}\} \frac{1}{\sigma\sqrt{2\pi}} \int \exp\{\frac{(t-h/2)^2}{2\sigma^2}\}dt \\ &= \exp\{-\frac{-h^2}{8\sigma^2}\}\end{aligned}$$

By an application of Hellinger's Theorem (2.1), we have (in any case) that

$$||\gamma - \gamma_h|| \geq 2\left(1 - \exp\{-\frac{-h^2}{8\sigma^2}\}\right) \tag{4}$$

Returning to $\mathcal{X}$, for every $f \in \mathcal{X}^*$, using the definition we find that $\gamma_h \circ f^{-1} = (\gamma \circ f^{-1})_{f(h)}$, and

$$||\gamma \circ f^{-1} - (\gamma \circ f^{-1})_{f(h)}||_{\mathbb{R}} \leq ||\gamma - \gamma_h||_{\mathcal{X}} \tag{5}$$

If $h \notin H$, there exists a sequence $(f_n) \subseteq \mathcal{X}^*$ with $||I^* f_n||_{\mathbb{L}_2(\mathcal{X},\gamma)} = 1$ and $f_n(h) \geq n$. By (4), and (5) we obtain

[2] Theorem 14.17 in [Jan97]

$$||\gamma - \gamma_h||_{\mathcal{X}} \geq ||\gamma \circ f^{-1} - (\gamma \circ f^{-1})_{f(h)}||_{\mathbb{R}} \geq 2\left(1 - \exp\{-\frac{1}{8} f_n(h)^2\}\right) \geq 2\left(1 - \exp\{-\frac{1}{8} n^2\}\right)$$

This implies that $||\gamma - \gamma_h|| = 2$ and by Lemma (2.2) we conclude they are mutually singular.

□

It follows from the theorem that for Gaussian measures every admissible shift defines an admissible direction.

Example 2.2. *Let $\mathcal{X} = C[0,1]$ and let W be a Wiener process with measure γ, and CM space H, recall from example (1.1) that*

$$\begin{aligned} H =& \{h \in AC[0,1], h(0) = 0, h' \in \mathbb{L}_2[0,1]\} \\ |h|_H^2 =& \int_0^1 h'(s) ds \ , \quad h \in H \end{aligned}$$

As for the functional Z associated to h by the formula $Iz = h$, we have that it coincides with the Wiener integral:

$$\begin{aligned} Iz(t) =& \delta_t(Iz) = (I * \delta_t, z) \\ =& \mathbb{E}W(t)z(W) = \left(\int_0^1 \mathbb{1}_{[0,t]} s dw(s) \cdot \int_0^1 h'(s) dw(s)\right) \\ =& \int_0^t h'(s) ds \\ =& h(t) \end{aligned}$$

Thus, $z(w) = \int_0^1 h'(s) dw(s)$, and the Cameron-Martin formula reads as

$$\frac{d\gamma_h}{d\gamma}(w) = \exp\{\int_0^1 h'(s) dw(s) - \frac{1}{2}\int_0^1 h'(s)^2 ds\}$$

This is the case considered by Cameron and Martin in [CM44]. For yet another proof of this case, see Theorem 19.23 [Kal21].

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