GENERICITY OF SPACES WITH THE EXTENDED GARCÍA-FALSET COEFFICIENT: $R(t, X) < 1 + t$ FOR SOME $t > 0$.

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Abstract. We state the classic result by [FZZ82] about genericity and extend it to include other geometric notions and consider a more general concept than that of a residual set. Moreover, we generalize two theorems in [DBP10] and [DBP08] that quantify the size of all renormings of very general classes of spaces that satisfy FPP.

We say that a Banach space $(X, ||\cdot||)$ satisfies the Fixed Point Property (FPP) if every 1-Lipschitz function under a non-empty closed, convex and weakly (w)-compact set has a fixed point. We ask ourselves: How many equivalent renormings of a space satisfy FPP? In order to address this question, we will consider the family of all possible renormings of a Banach space and analyze how many of these satisfy FPP. We would like to answer that *almost all* renormings preserve FPP.

1. Genericity and FPP

In general, we will say that property (P) is *generic* in a set A if all elements of A satisfy property (P), but in a 'small set', and we will say that A satisfies (P) 'almost always'.

The first step is to precise what do we mean by a 'small set'. Depending on the framework we will be working with, there can be many instances of a null or negligible set. In the context of cardinality, a small set is a countable one. However, this is a poor class lacking interesting properties. In a measure space (X, Σ, μ), negligible sets are those precisely to be μ-null sets. A problem with this definition is the difficulty to extend Lebesgue measure to infinite dimensional spaces.

A more proper notion which we will work with, is the topological one:

Definition 1.1 (Meagre set)**.** *We will sat that $A \subseteq X$ a subset of a topological space is meagre (or of the first category) if it is the countable union of nowhere dense sets, this is sets whose closure has empty interior.*

We will call the complement of a meagre set, this is the countable intersection of sets with dense interior, a *residual set.*

Remark 1.1. *It is important to note that all notions of 'small' set mentioned above formed a σ-ideal, this is, a class closed under countable unions and under set inclusion.*

We will work with the family of all possible renormings of a Banach space. It is known that ℓ_1 and $\mathbb{L}_1$ fail to have FPP. However, this spaces can be renormed so that they satisfy FPP, so that this property is not preserved under isomorphisms. Inspired by the negative results of [DLT96] (ℓ_∞, $\ell_1(\Gamma)$ and $c_0(\Gamma)$) fail FPP under every equivalent renorming, so that we will restrict ourselves to work with reflexive spaces.

It is possible to prove that reflexive spaces can be renormed to satisfy FPP. The extension of this result in the framework of genericity is the following: How many renormings do satisfy FPP? In [FZZ82] an answer is provided in the separable case. The authors prove that almost all renorming, in the categorical sense, of a *uniformly convex in every direction (UCED)* space is still UCED. Due to Zizler result [Ziz71], every separable Banach space admits an UCED norm. As this property ultimately implies FPP, we obtain the following corollary: If X is a reflexive, and separable Banach space, then almost all renormings of X satisfy FPP.

We will denote by $\mathcal{P}$ to be the family of all equivalent renormings of the Banach space $(X, ||\cdot||)$. We will endow the space $\mathcal{P}$ with the metric

$$\rho(p,q) = \sup\{|p(x) - q(x)| \mid x \in B_X\}$$

so that the resulting metric space $(\mathcal{P}, \rho)$ is a Baire space.

The first result about genericity in this context is given by the following theorem.

Theorem 1.1 ([FZZ82])**.** *Let (X, r) be a UCED space. Then there is a residual subset $\mathcal{R}$ (moreover a dense G_δ) of $\mathcal{P}$, such that for all $p \in \mathcal{R}$, the space (X, p) is UCED.*

A careful reading of the proof of theorem (1.1), yields the following remark, which we state in form of a lemma.

We will say that a norm r is P-regular if for any two sequences $\{x_n\}, \{y_n\} \subseteq X$ such that $r(x_n), r(y_n) \leq 1/2$ for all n and

$$\lim 2r^2(x_n) + 2r^2(y_n) - r^2(x_n + y_n) = 0$$

then $x_n - y_n \longrightarrow 0$ under some topology.

Lemma 1.2. *Let (X, r) be a P-regular space and $q \in \mathcal{P}$ a norm such that for all $k \in \mathbb{N}$ there exists $p_k \in \mathcal{P}$ y $j_k \geq k$ such that*

$$\rho(q, (\frac{r^2}{j_k} + p_k^2)^{\frac{1}{2}}) < \mathcal{O}(\frac{1}{j_k})$$

then q is P-regular.

Proof. Let $\{x_n\}, \{y_n\} \subseteq X$ be two sequences such that $q(x_n), q(y_n) \leq 1/2$ and

$$\lim 2q^2(x_n) + 2q^2(y_n) - q^2(x_n + y_n) = 0$$

Set $k \in \mathbb{N}$. There exists $p_k \in \mathcal{P}$ y $j_k \geq k$ with $\rho(q, (\frac{r^2}{j_k} + p_k^2)^{\frac{1}{2}}) < \mathcal{O}(\frac{1}{j_k})$

Note that by the equivalence of norms, we can scale the sequences in order that $r(x_n), r(y_n) \leq 1/2$ for all n. By the convexity of p_k

$$\begin{aligned}\frac{1}{j_k}(2r^2(x_n) + 2r^2(y_n) - r^2(x_n + y_n)) &\leq 2(p_k^2 + \frac{1}{j_k}r^2)(x_n) + 2(p_k^2 + \frac{1}{j_k}r^2)(y_n) \\ &\quad - (p_k^2 + \frac{1}{j_k}r^2)(x_n + y_n) \\ &\leq \mathcal{O}(\frac{1}{j_k}) + (2q^2(x_n) + 2q^2(y_n) - q^2(x_n + y_n))\end{aligned}$$

and then,

$$\begin{aligned}&\limsup \frac{1}{j_k}(2r^2(x_n) + 2r^2(y_n) - r^2(x_n + y_n)) \\ &\leq \mathcal{O}(\frac{1}{j_k}) + \limsup(2q^2(x_n) + 2q^2(y_n) - q^2(x_n + y_n)) \\ &= \mathcal{O}(\frac{1}{j_k})\end{aligned}$$

as k is arbitrary we can conclude that

$$\limsup \frac{1}{j_k}(2r^2(x_n) + 2r^2(y_n) - r^2(x_n + y_n)) = 0$$

Finally, by P-regularity of the norm r we can conclude that $x_n - y_n \longrightarrow 0$ under some topology, so that q is P-regular.

□

Intuitively the lemma tells us that sufficiently close norm to a P-regular norm, is itself P-regular. In other words, P-regular norms are stable under 'small' perturbations.

With the previous lemma, we can state Theorem (1.1) under more general convexity notions.

Theorem 1.3. *Let (X, r) be a Banach space with r having a P-regular geometric property. Then there is a residual subset $\mathcal{R}$ of $\mathcal{P}$, such that for all $p \in \mathcal{R}$, the space (X, p) is P-regular.*

2. Main results

We shall emphasize that the categorical and the measure-theoretic notions of negligible sets can be different when both coexists. For example, the real line $\mathbb{R}$ can be decomposed as the disjoint union of a measure-zero set and a set of the first category.

In order to avoid such conflicts, we will use the concept of *porosity* of a set. Which can be seen to generalize the notions of measure-zero and meagre set.

Definition 2.1 (Porosity)**.** *Given a Banach space $(X, ||\cdot||)$, a set $M \subseteq X$ is called* porous *if for all $x \in M$, there is $\Delta \in (0,1)$ such that for all $r > 0$ there exists $\widetilde{x} \in X$ such that $||\widetilde{x} - x|| \leq r$ and $B(\widetilde{x}, \Delta\, ||\widetilde{x} - x||) \cap M = \emptyset$. We will called the countable union of porous sets to be σ-porous.*

Remark 2.1. *A sufficient condition to prove porosity of a set $M \subseteq X$ is that for all $x \in M$ there exists $\Delta > 0$ and $r_0 > 0$ such that for all $r \in (0, r_0]$, there is $\widetilde{x} \in X$ with*

$$B(\widetilde{x}, \Delta r) \subset B(x, r) \setminus M$$

For a detailed survey about porous sets and its properties we refer to [Zaj05] and references therein.

It is possible to reformulate Theorem (1.1) with the notion of porosity.

Theorem 2.1. *Let X be a Banach P-regular, separable and reflexive space. Then there is a σ-porous set $\mathcal{R} \subseteq \mathcal{P}$ such that for all $p \in \mathcal{P} \setminus \mathcal{R}$, (X, p) is P-regular.*

Proof. As the space X is P-regular, there exists a P-regular norm, r. For an arbitrary $p \in \mathcal{P}$, let $m(p) = \inf_{r(x)=1} p(x)$ and $p_j = \sqrt{p^2 + (r^2/j)}$. For all $x \in S_{(X,r)}$

$$|p_j(x) - p(x)| = \frac{|p_j^2(x) - p^2(x)|}{p_j(x) + p(x)} \leq \frac{r^2(x)}{jp(x)} \leq \frac{1}{jm(p)}$$

so that $\rho(p, p_j) \leq \frac{1}{jm(p)}$.

Define

$$A_n = \left\{ p \in \mathcal{P} \mid \frac{1}{n} \leq m(p) \right\}$$

and

$$G_k = \bigcup_{p \in \mathcal{P}} \bigcup_{j \geq k} B\left(p_j, \frac{1}{kj}\right)$$

We claim that $A_n \setminus G_k$ is porous with $r_0 = 1/k$ and $\Delta = \frac{1}{4kn}$.

Let $s < 1/k$, so that $\frac{2n}{s} \geq \frac{1}{s} > k \geq 1$. This implies there is an integer $j \geq k$ with $j \in (\frac{2n}{s}, \frac{4n}{s})$ and then $\frac{s}{4n} \leq \frac{1}{j} < \frac{s}{2n}$. Let $p \in A_n \setminus G_k$. Si $q \in B(p_j, \frac{s}{4kn}) = B(p_j, s\Delta)$, then

$$\begin{aligned}\rho(p,q) \leq& \rho(p,p_j) + \rho(p_j,q)\\ \leq& \frac{1}{jm(p)} + \frac{s}{4kn}\\ \leq& \frac{n}{j} + \frac{s}{4kn}\\ \leq& \frac{s}{2} + \frac{s}{2} = s\end{aligned}$$

Then $B(p_j, s\Delta) \subset B(p,s)$, moreover as $B(p_j, \frac{1}{kj}) \in G_k$ and $\frac{s}{4kn} < \frac{1}{kj}$, the ball $B(p_j, s\Delta)$ does not meet $A_n \setminus G_k$ and therefore

$$\mathcal{R} = \cup_{n,k} A_n \setminus G_k$$

is σ-porous.

We will conclude the proof of the Theorem once we establish that all $q \in \mathcal{P} \setminus \mathcal{R}$ is P-regular. Let $q \in \mathcal{P} \setminus \mathcal{R}$ and note that

$$\mathcal{R} = \cup_{n,k} A_n \setminus G_k = \cup_k (\mathcal{P} \setminus G_k) = \mathcal{P} \setminus \cap_k G_k$$

this implies that $\mathcal{P} \setminus \mathcal{R} = \cap_k G_k$. Therefore $q \in \cap_k G_k$, so that it exists $p = p(k) \in \mathcal{P}$ and $j \geq k$ with $q \in B(p_j, \frac{1}{kj})$. As the hypothesis of lemma 1.2 are established, we conclude that q is P-regular.

□

Notice that this arguments are not easily extended to the non-separable case [KT82]. Recall the following geometric coefficient introduced in [GF97].

$$R(X) := \sup\{\lim_{\mathcal{U}} ||x_n - x||\}$$

where the supremum is taken over all weakly null-sequences and all $x \in B_X$.

It is interesting to address the question whether FPP is a generic property. The answer is affirmative in the case where $R(X) < 2$: In [DBP08] it is proved that if a Banach space X satisfies that $R(X) < 2$, then all equivalent renormings ρ satisfy $R(X;\rho) < 2$ (so that they satisfy FPP), except in a σ-porous set.

Theorem 2.2 ([DBP08]). *Let X be a Banach space with $R(X) < 2$. Then there is a σ-porous set $\mathcal{R} \subseteq \mathcal{P}$ such that for any $p \in \mathcal{P} \setminus \mathcal{R}$, it is true that $R(X;p) < 2$.*

In this paper we extend this result to spaces whose extended geometrical coefficient satisfy $R(t,X) < 1+t$ for some $t > 0$. Where the coefficient $R(t,X)$ is defined as in [DB96].

Definition 2.2. *Let X be a Banach space. For any non-negative number t and a non-trivial ultrafilter $\mathcal{U}$ over $\mathbb{N}$, we define the extended geometrical coefficient as*

$$R(t,X) = \sup\{\lim_{\mathcal{U}} ||x + x_n|| \mid ||x|| \leq t\,, ||x_n|| \leq 1\, \forall n \;\; \text{and} \;\; x_n \xrightarrow{w} 0 \;\text{with}\; \lim_{\mathcal{U},m} \lim_{\mathcal{U},n\neq m} ||x_n - x_m|| \leq 1\}$$

Next, we will present two key lemmas. This results are concerned with the stability of the geometrical coefficient, under small perturbations of the norm that underlies the geometry of the space.

When there is no confusion regrading the space X we are working with, we will denote the coefficient $R(t,(X,q))$ as $R(t,q)$, with the purpose of keeping notation as simple as possible.

Lemma 2.3. *Let $(X, ||\cdot||)$ be a Banach space and $p \in \mathcal{P}$ with*

$$m_t(p) = \inf_{x \in tB_{(X,||\cdot||)}} p(x) \qquad M_t(p) = \sup_{x \in tB_{(X,||\cdot||)}} p(x)$$

Let $\lambda \in (0,1)$ and $p_\lambda(x) = p(x) + \lambda \|x\|$. Then

$$R(t,p_\lambda) \leq (1+t) - \left[\frac{\lambda}{\frac{M_t(p)}{t}+\lambda}\right]((1+t) - R(t, \|\cdot\|))$$

Proof. Let $\{x_n\} \subset X$ be a w-null sequence with $\lim_{m\neq n} p_\lambda(x_n - x_m) \leq 1$,

$\lim_{m\neq n} \|x_n - x_m\| \leq 1$ (possible due to the equivalence of norms) and $x \in X$ such that $p_\lambda(x_n) \leq 1$, $p_\lambda(x_n) \to 1$ and $p_\lambda(x) = t$. Without loss of generality, suppose that $\|x_n\| \to a$ and $\|x\| = b$; so that $p(x_n) \to 1 - \lambda a$ and $p(x) = t - \lambda b$.

Suppose further that $b - at \geq 0$. Note then that

$$\begin{aligned}\lim \|x_n + x\| \leq& \lim \left(\left\| a(\frac{x_n}{a} + \frac{tx}{b}) \right\| + \left\| x - \frac{at}{b}x \right\| \right) \\ \leq& aR(t, \|\cdot\|) + b - at\end{aligned}$$

and then

$$\begin{aligned}\lim p(x_n + x) + \lambda \|x_n + x\| \leq& \lim p(x_n) + p(x) + \lambda \|x_n + x\| \\ \leq& 1 - \lambda a + t - \lambda b + \lambda(aR(t, \|\cdot\|) + b - at) \\ =& (1+t) - \lambda a ((1+t) - R(t, \|\cdot\|)) \qquad (1)\end{aligned}$$

If now, $b - at < 0$:

$$\begin{aligned}\lim \|x_n + x\| \leq& \lim \left(\left\| \frac{b}{t}(\frac{x_n}{a} + \frac{tx}{b}) \right\| + \left\| x_n - \frac{b}{at}x_n \right\| \right) \\ \leq& \frac{b}{t}R(t, \|\cdot\|) + a - \frac{b}{t}\end{aligned}$$

and

$$\begin{aligned}\lim p(x_n + x) + \lambda \|x_n + x\| \leq& \lim p(x_n) + p(x) + \lambda \|x_n + x\| \\ \leq& 1 - \lambda a + t - \lambda b + \lambda(\frac{b}{t}R(t, \|\cdot\|) + a - \frac{b}{t}) \\ =& (1+t) - \lambda\frac{b}{t} ((1+t) - R(t, \|\cdot\|)) \qquad (2)\end{aligned}$$

As $p(y) \leq \frac{M_t(p)\|y\|}{t}$ for all $y \in X$, then $p(y) + \lambda \|y\| \leq \left(\frac{M_t(p)}{t} + \lambda\right) \|y\|$ and then

$$\|y\| \geq \frac{p_\lambda(y)}{\left(\frac{M_t(p)}{t} + \lambda\right)}$$

We can conclude now that

$$\frac{1}{\left(\frac{M_t(p)}{t} + \lambda\right)} = \lim \frac{p_\lambda(x_n)}{\left(\frac{M_t(p)}{t} + \lambda\right)} \leq \lim \|x_n\| = a$$

and

$$\frac{1}{\left(\frac{M_t(p)}{t} + \lambda\right)} = \frac{p_\lambda(x)/t}{\left(\frac{M_t(p)}{t} + \lambda\right)} \leq \frac{\|x\|}{t} = \frac{b}{t}$$

From this inequalities and (1, 2) we obtain the desired inequality.

□

Lemma 2.4. *Let $p, q \in \mathcal{P}$ be such that $\rho(p,q) < \epsilon$. Then*

$$R(t,q) \leq \frac{m_t(p)R(t,p) + \epsilon t(1+t)}{m_t(p) - \epsilon t}$$

Proof. Let $\{x_n\} \subset X$ a w-null sequence such that $\lim_{m\neq n} q(x_n - x_m) \leq 1$,

$\lim_{m\neq n} p(x_n - x_m) \leq 1$ and $x \in X$ with $q(x_n) \leq 1$, $q(x_n) \to 1$ y $q(x) = t$. As $p(x) \leq q(x) + \epsilon\,||x||$ for all $x \in X$; then $p(x) - \epsilon t \leq q(x)$ as long as $||x|| = t$. Then $\frac{p(x)}{q(x)} \leq \frac{p(x)}{p(x)-\epsilon t}$ and as the function $r \longmapsto \frac{r}{r-\epsilon t}$ is decreasing and $p(x) \geq m_t(p)$ for all $||x|| = t$:

$$p(x) \leq \frac{p(x)}{p(x) - \epsilon t} q(x) \leq \frac{m_t(p)}{m_t(p) - \epsilon t} q(x)$$

This implies that

$$p\left(\frac{m_t(p) - \epsilon t}{m_t(p)} y\right) \leq a \qquad \text{as long as} \quad q(y) \leq a$$

Then,

$$p\left(\frac{m_t(p) - \epsilon t}{m_t(p)} x_n\right) \leq 1 \qquad p\left(\frac{m_t(p) - \epsilon t}{m_t(p)} x\right) \leq t$$

Moreover, as

$$m_t(p) \leq p\left(\frac{tx}{||x||}\right) \quad \forall\, x \in X$$

then

$$||x|| \leq \frac{p(x)t}{m_t(p)}$$

Therefore

$$\begin{aligned}
\lim q(x_n + x) &\leq \lim p(x_n + x) + \epsilon\,||x_n + x|| \\
&= \lim \left[\frac{m_t(p)}{m_t(p) - \epsilon t}\right] p\left(\frac{m_t(p) - \epsilon t}{m_t(p)}(x_n + x)\right) + \epsilon\,||x_n + x|| \\
&\leq \frac{m_t(p)}{m_t(p) - \epsilon t} R(t,p) + \epsilon t \frac{p(x_n + x)}{m_t(p)} \\
&\leq \frac{m_t(p)}{m_t(p) - \epsilon t} R(t,p) + \epsilon t \frac{q(x_n + x)}{m_t(p) - \epsilon t} \\
&\leq \frac{m_t(p)R(t,p) + \epsilon t(1+t)}{m_t(p) - \epsilon t}
\end{aligned}$$

□

We are ready to prove and state our generalization.

Theorem 2.5. *Let $(X, ||\cdot||)$ a Banach space satisfying $R(t, ||\cdot||) < 1+t$. Then there is a σ-porous set $\mathcal{R} \subseteq \mathcal{P}$ such that for all $p \in \mathcal{P} \setminus \mathcal{R}$, then $R(t,p) < 1+t$, and so (X,p) has FPP.*

Proof. Let

$$B_n = \{p \in \mathcal{P} \mid \frac{t}{n} < m_t(p) < M_t(p) < tn\}$$

and

$$A_n = B_n \setminus \bigcup_{\lambda\in(0,1),\, p\in\mathcal{P}} B\left(p_\lambda, \frac{(1+t) - R(t,||\cdot||)}{(1+t)^b n(n+2)}\lambda\right)$$

with $b = \frac{\log 2}{\log 1+t} + 2$. The claim is that A_n is porous. We will use as witness for the porosity $\Delta = \frac{(1+t)-R(t,||\cdot||)}{2(1+t)^b n(n+2)}$ y $r_0 = 1$.

Consider $r \in (0,1)$ and $p \in \mathcal{P}$. Let $\lambda = \frac{r}{2}$ and note that for all $x \in B_X$,

$$|p(x) - p_{\frac{r}{2}}(x)| = \frac{r}{2}\,||x|| \leq \frac{r}{2}$$

Then $\rho(p, p_{\frac{r}{2}}) \leq \frac{r}{2}$. If $q \in B(p_{\frac{r}{2}}, \Delta r)$, then

$$\rho(p,q) \leq \rho(p, p_{\frac{r}{2}}) + \rho(p_{\frac{r}{2}}, q) \leq \frac{r}{2} + \Delta r \leq \frac{r}{2} + \frac{r}{2} = r$$

So that the ball $B(p_{\frac{r}{2}}, \Delta r)$ is entirely contained in $B(p, r)$, and it is not contained in A_n because $B(p_{\frac{r}{2}}, \Delta r) = B\left(p_\lambda, \frac{(1+t)-R(t,||\cdot||)}{(1+t)^b n(n+2)}\lambda\right)$. We can infer then that $\mathcal{R} = \cup_n A_n$ is σ-porous.

We will conclude the proof once we set that for any $q \in \mathcal{P} \setminus \mathcal{R}$, then $R(t, q) < 1 + t$. Let $q \in \mathcal{P} \setminus \mathcal{R}$. Note that $\mathcal{P} = \cup_n B_n$ and so for some n

$$\frac{t}{n} < m_t(q) < M_t(q) < tn$$

Note that, as $q \in B_n$ and $q \notin A_n$ for all n, then $q \in B\left(p_\lambda, \frac{(1+t)-R(t,||\cdot||)}{(1+t)^b n(n+2)}\lambda\right)$ for some $p \in \mathcal{P}$ and $\lambda \in (0, 1)$, so that $q \in A_n$. Therefore

$$\begin{aligned}
\rho(q, p_\lambda) \leq & \frac{(1+t) - R(t, ||\cdot||)}{(1+t)^b n(n+2)}\lambda \\
\leq & \frac{(1+t) - R(t, ||\cdot||)}{(1+t)^b n(\frac{M_t(q)}{t} + 2)}\lambda \\
\leq & \frac{(1+t) - R(t, ||\cdot||)}{(1+t)^b n(\frac{M_t(p)}{t} + 1)}\lambda \\
\leq & \frac{(1+t) - R(t, ||\cdot||)}{(1+t)^b n(\frac{M_t(p)}{t} + \lambda)}\lambda \\
\overset{2.3}{\leq} & \frac{(1+t) - R(t, p_\lambda)}{(1+t)^b n} \\
\leq & \frac{[(1+t) - R(t, p_\lambda)]\,(\frac{m_t(p)}{t} + \lambda)}{(1+t)^b} := \epsilon
\end{aligned}$$

by the lemma (2.4), we obtain that

$$\begin{aligned}
R(t, q) \leq & \frac{m_t(p_\lambda)R(t, p_\lambda) + \epsilon t(1+t)}{m_t(p_\lambda) - \epsilon t} \\
= & \frac{m_t(p_\lambda)R(t, p_\lambda) + (1+t)\left\{\frac{[(1+t)-R(t,p_\lambda)](m_t(p)+\lambda t)}{(1+t)^b}\right\}}{m_t(p_\lambda) - \left\{\frac{[(1+t)-R(t,p_\lambda)](m_t(p)+\lambda t)}{(1+t)^b}\right\}}
\end{aligned}$$

and noting that $m_t(p) + \lambda t = m_t(p_\lambda)$

$$\begin{aligned}
R(t, q) \leq & \frac{(1+t)^b R(t, p_\lambda)m_t(p_\lambda) + (1+t)^2 m_t(p_\lambda) - (1+t)R(t, p_\lambda)m_t(p_\lambda)}{(1+t)^b m_t(p_\lambda) - (1+t)m_t(p_\lambda) + R(t, p_\lambda)m_t(p_\lambda)} \\
= & \frac{\left[(1+t)^b - (1+t)\right] R(t, p_\lambda) + (1+t)^2}{R(t, p_\lambda) + [(1+t)^b - (1+t)]}
\end{aligned}$$

It can be proves that $\frac{\left[(1+t)^b-(1+t)\right]R(t,p_\lambda)+(1+t)^2}{R(t,p_\lambda)+[(1+t)^b-(1+t)]} < 1+t$ as long as $R(t, p_\lambda) < 1+t$; which is true by lemma (2.3) and finally $R(t, q) < 1 + t$.

□

One of the main tools to find an equivalent norm satisfying FPP is the existence of an immersion to $c_0(\Gamma)$ (this property is satisfied by very general spaces as spaces with a Markushevich basis or weakly compactly generated spaces, etc). The next theorem generalizes the previous claim, and from it we can conclude that ***almost all*** renormings of a space with an unconditional basis, or reflexive spaces, etc. satisfy FPP.

Theorem 2.6 ([DBP10])**.** *Let X be a Banach space. Suppose there is a one-to-one linear bounded operator $J : X \longmapsto Y$, where $R(Y) < 2$. Then there is a residual set $\mathcal{R} \subseteq \mathcal{P}$ such that for all $p \in \mathcal{R}$, the space (X, p) satisfies FPP.*

We generalize this result in two directions: (1) for spaces which are embedded to a space Y with $R(t, Y) < 1 + t$ for some $t > 0$ and (2) quantify the size of all renormings with FPP to be the complement of a σ-porous set.

Theorem 2.7. *Let X be a Banach space. Suppose there is a one-to-one linear bounded operator $J : X \longmapsto Y$, where $R(t, Y) < 1 + t$ for some $t > 0$. Then there is a σ-porous set $\mathcal{R} \subseteq \mathcal{P}$ such that for all $p \in \mathcal{P} \setminus \mathcal{R}$, then $R(t, (X, p)) < 1 + t$, and so (X, p) has FPP.*

The theorem is now a simple corollary of 2.5 and the following lemma, whose proof is analogous to the proof in 2.3.

Lemma 2.8. *Let $(X, ||\cdot||_X)$ be a Banach space. Suppose there is a one-to-one linear bounded operator $J : X \longmapsto Y$, where $R(t, Y) < 1+t$ for some $t > 0$. Then there is a renorming p such that $R(t', (X, p)) < 1+t'$ with $t' > 0$.*

Proof. Note that an equivalent norm of X is given by

$$p_\lambda(x) = ||x||_X + \lambda \, ||Jx||_Y$$

In a completely analogous fashion as in lemma (2.3), it is possible to prove that $R(t', p_\lambda) < 1 + t'$ for some $t' > 0$. ☐

3. Conclusion and further problems

We conclude the paper with some open problems.

(i) Is FPP a generic property? This is, if X satisfies with FPP, then does *almost every* renorming satisfies FPP as well?

Even if FPP does not result in a generic property, then for which classes of spaces FPP is generic? In this paper we have proves that for reflexive spaces FPP is generic.

In [PT95], an infinite dimensional space is decomposed into a Haar-null set and a porous set. This justifies the following question.

(ii) Is it possible to consider other notions of a *small set*? For example, consider **Haar-null**[1] sets , **Haar-meagre**[2] sets, or **HP-small**[3] sets.

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[1] For a definition see [HSY92].
[2] For a definition see [Dar13].
[3] For a definition and connection with Haar-null and σ-porous sets see [Kol01].

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